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Eigen elastic mechanics and its variation principle①

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[Abstract] The fundamental equations and the corresponding boundary condition of elastic mechanics under mechanical representation are given by using the conception of eigen space and elastic variation principle. It is proved theoretically that the solution of anisotropic elastic mechanics consists of modal ones, which are obtained respectively from the modal equation of the different subspaces. A simple application is also given.

[Key words] anisotropy; eigen elastic mechanics; variation principle; modal equation
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1 INTRODUCTION

The conception of eigen elastic mechanics benefits from the standard space theory[1~4]. A series of studies[5~7] made by author recently, have established the eigen form of elastic mechanics under mechanical representation and its operationalized principle, but the corresponding boundary condition and the solving method have not been discussed in more detail yet. It is well known that a correct boundary equation for elastic body can be obtained only by means of the variation principle. But the classical variation principle of elastic mechanics is a energy expression of the equilibrium equation and the boundary condition of elastic body under geometrical representation. If we want to study the variation principle of elastic body under mechanical representation, it is necessary to convert the mechanical quantities, such as stress or strain, from the geometrical space to the mechanical one. This is just the work to be made in this paper. Furthermore, a definite, explicit and complete scheme to solve the anisotropic elastic mechanics is also given.

2 CONCEPTION OF REPRESENTATION CONVERSION

The relationship of the representation conversion of stress and strain vectors[8~10] are respectively

$$\sigma^* = \Phi^T \cdot \sigma, \quad \varepsilon^* = \Phi^T \cdot \varepsilon \tag{1}$$

and it also can be rewritten as follows.

$$\left.\begin{aligned} \sigma_x &= \sum \sigma_i^* \varphi_{i1}, \quad \sigma_y = \sum \sigma_i^* \varphi_{i2}, \quad \sigma_z = \sum \sigma_i^* \varphi_{i3} \\ \tau_{zy} &= \sum \sigma_i^* \varphi_{i4}, \quad \tau_{zx} = \sum \sigma_i^* \varphi_{i5}, \quad \tau_{xy} = \sum \sigma_i^* \varphi_{i6} \end{aligned}\right\} \tag{2}$$

where φ_{ij} is a value of jth element in the ith modal vector.

3 EIGEN ELASTIC VARIATION PRINCIPLE

Considering a elastic body subjected to a constant loading under equilibrium, the displacement is known in S_u part of the boundary and the surface force is known in S_σ part of the boundary. Supposing that the body force is denoted as f_x, f_y, f_z, the surface force X_s, Y_s, Z_s, and the fictitious one of the displacement u, v, w at any point of elastic body, δu, δv, δw. According to the fictitious displacement principle,

$$\delta V = \iiint (f_x \delta u + f_y \delta v + f_z \delta w) \mathrm{d}x \mathrm{d}y \mathrm{d}z + \iint_S (X_s \delta u + Y_s \delta v + Z_s \delta w) \mathrm{d}S \tag{3}$$

Because the displacement is known in S_u part of the boundary, the fictitious displacement hold $\delta u = \delta v = \delta w = 0$ in S_u, so Eqn.(3) becomes

$$\delta [V - \iiint (f_x u + f_y v + f_z w) \mathrm{d}x \mathrm{d}y \mathrm{d}z - \iint_{S_\sigma} (X_s u + Y_s v + Z_s w) \mathrm{d}S] = 0 \tag{4}$$

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This is the minimal potential principle. It is equal to three equilibrium equations and three boundary conditions of static force. Under mechanical representation, the strain energy of elastic body is defined as

$$\delta V = \iiint \delta W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{5}$$

Using the representation conversion relationship, Eqn. (1) and the orthogonality of the modal matrix, $\Phi^{\mathrm{T}} \cdot \Phi = \boldsymbol{I}$, the variation of strain energy of elastic body in unit volume becomes

$$\delta W = \sigma^{\mathrm{T}} \cdot \delta\varepsilon = \sigma^{*\mathrm{T}} \Phi^{\mathrm{T}} \Phi \delta\varepsilon^{*} = \sigma^{*\mathrm{T}} \cdot \delta\varepsilon^{*} = \sum \sigma_i^{*} \delta\varepsilon_i^{*} \tag{6}$$

From the geometrical relationship of elastic body,

$$\varepsilon = \{ \partial u/\partial x,\ \partial v/\partial y,\ \partial w/\partial z,\ \partial w/\partial y + \partial v/\partial z,\ \partial u/\partial z + \partial w/\partial x,\ \partial u/\partial y + \partial v/\partial x \}^{\mathrm{T}} \tag{7}$$

Using Eqns. (5) and (6), the variation of strain energy of elastic body becomes

$$\delta V = \iiint \sum_i \sigma_i^{*} \varphi_i^{\mathrm{T}} \delta\varepsilon] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint \sum_i \sigma_i^{*} (\sum_j \varphi_{ij} \delta\xi_j)] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sum_i \sum_j \iiint \sigma_i^{*} \varphi_{ij} \delta\xi_j] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{8}$$

Using Eqn. (7), Eqn. (8) becomes

$$\delta V = \sum \iiint \left\{ \sigma_i^{*} [\varphi_{i1} \frac{\partial}{\partial x}\delta u + \varphi_{i2}\frac{\partial}{\partial y}\delta v + \varphi_{i3}\frac{\partial}{\partial z}\delta w + \varphi_{i4}(\frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial z}\delta v) + \varphi_{i5}(\frac{\partial}{\partial z}\delta u + \frac{\partial}{\partial x}\delta w) + \varphi_{i6}(\frac{\partial}{\partial y}\delta u + \frac{\partial}{\partial x}\delta v)] \right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{9}$$

Integrating Eqn. (9) separately,

$$\left.\begin{aligned}
&\iiint \sum \sigma_i^{*} \varphi_{i1}) \frac{\partial}{\partial x}\delta u \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_\sigma} (\sum \sigma_i^{*} \varphi_{i1}) l \delta u \,\mathrm{d}S - \iiint \frac{\partial(\sum \sigma_i^{*} \varphi_{i1})}{\partial x} \delta u \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \\
&\iiint \sum \sigma_i^{*} \varphi_{i2}) \frac{\partial}{\partial y}\delta v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_\sigma} (\sum \sigma_i^{*} \varphi_{i2}) m \delta v \,\mathrm{d}S - \iiint \frac{\partial(\sum \sigma_i^{*} \varphi_{i2})}{\partial y} \delta v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \\
&\iiint \sum \sigma_i^{*} \varphi_{i3}) \frac{\partial}{\partial z}\delta w \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_\sigma} (\sum \sigma_i^{*} \varphi_{i3}) n \delta w \,\mathrm{d}S - \iiint \frac{\partial(\sum \sigma_i^{*} \varphi_{i3})}{\partial z} \delta w \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\iiint \sum \sigma_i^{*} \varphi_{i4}) \left[\frac{\partial}{\partial y}\delta w + \frac{\partial}{\partial z}\delta v \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_\sigma} (\sum \sigma_i^{*} \varphi_{i4}) (m\delta w + n \delta v) \mathrm{d}S - \\
&\qquad \iiint \left[\frac{\partial(\sum \sigma_i^{*} \varphi_{i4})}{\partial y} \delta w + \frac{\partial(\sum \sigma_i^{*} \varphi_{i4})}{\partial z} \delta v \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \\
&\iiint \sum \sigma_i^{*} \varphi_{i5}) \left[\frac{\partial}{\partial z}\delta u + \frac{\partial}{\partial x}\delta w \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_\sigma} (\sum \sigma_i^{*} \varphi_{i5}) (n\delta u + l \delta w) \mathrm{d}S - \\
&\qquad \iiint \left[\frac{\partial(\sum \sigma_i^{*} \varphi_{i5})}{\partial z} \delta u + \frac{\partial(\sum \sigma_i^{*} \varphi_{i5})}{\partial x} \delta w \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \\
&\iiint \sum \sigma_i^{*} \varphi_{i6}) \left[\frac{\partial}{\partial y}\delta u + \frac{\partial}{\partial x}\delta v \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_{S_\sigma} (\sum \sigma_i^{*} \varphi_{i6}) (l\delta v + m \delta u) \mathrm{d}S - \\
&\qquad \iiint \left[\frac{\partial(\sum \sigma_i^{*} \varphi_{i6})}{\partial x} \delta v + \frac{\partial(\sum \sigma_i^{*} \varphi_{i6})}{\partial y} \delta u \right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned}\right\} \tag{10}$$

where l, m, n are the direction cosine of normal N in S_σ part of the boundary.

Substituting Eqn. (10) into Eqn. (9),

$$\begin{aligned}
\delta V = & \iint_{S_\sigma} \{ [(\sum \sigma_i^{*} \varphi_{i1}) l + (\sum \sigma_i^{*} \varphi_{i6}) m + (\sum \sigma_i^{*} \varphi_{i5}) n] \delta u + [(\sum \sigma_i^{*} \varphi_{i6}) l + (\sum \sigma_i^{*} \varphi_{i2}) m + \\
& (\sum \sigma_i^{*} \varphi_{i4}) n] \delta v + [(\sum \sigma_i^{*} \varphi_{i5}) l + (\sum \sigma_i^{*} \varphi_{i4}) m + (\sum \sigma_i^{*} \varphi_{i3}) n] \delta w \} \mathrm{d}S - \\
& \iiint \{ [\frac{\partial(\sum \sigma_i^{*} \varphi_{i1})}{\partial x} + \frac{\partial(\sum \sigma_i^{*} \varphi_{i2})}{\partial y} + \frac{\partial(\sum \sigma_i^{*} \varphi_{i4})}{\partial z}] \delta u + [\frac{\partial(\sum \sigma_i^{*} \varphi_{i6})}{\partial x} + \frac{\partial(\sum \sigma_i^{*} \varphi_{i2})}{\partial y} + \\
& \frac{\partial(\sum \sigma_i^{*} \varphi_{i4})}{\partial z}] \delta v + [\frac{\partial(\sum \sigma_i^{*} \varphi_{i5})}{\partial x} + \frac{\partial(\sum \sigma_i^{*} \varphi_{i4})}{\partial y} + \frac{\partial(\sum \sigma_i^{*} \varphi_{i3})}{\partial z}] \delta w \} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z
\end{aligned} \tag{11}$$

Substituting Eqn. (11) into the minimal potential principle, Eqn. (4), and considering that the fictitious displacements δu, δv, δw have no any connection, and are totally arbitrary within elastic body and in the boundary where the external force is definite, we can get three equilibrium equations of elastic body and three boundary conditions of static force in the surface where external force is given under mechanical representation, when letting the coefficients of δu, δv and δw under integration be zero.

$$\left.\begin{aligned}\frac{\partial}{\partial x}\left(\sum\sigma_i^*\varphi_{i1}\right)+\frac{\partial}{\partial y}\left(\sum\sigma_i^*\varphi_{i6}\right)+\frac{\partial}{\partial z}\left(\sum\sigma_i^*\varphi_{i5}\right)+f_x=0\\\frac{\partial}{\partial x}\left(\sum\sigma_i^*\varphi_{i6}\right)+\frac{\partial}{\partial y}\left(\sum\sigma_i^*\varphi_{i2}\right)+\frac{\partial}{\partial z}\left(\sum\sigma_i^*\varphi_{i4}\right)+f_y=0\\\frac{\partial}{\partial x}\left(\sum\sigma_i^*\varphi_{i5}\right)+\frac{\partial}{\partial y}\left(\sum\sigma_i^*\varphi_{i4}\right)+\frac{\partial}{\partial z}\left(\sum\sigma_i^*\varphi_{i3}\right)+f_z=0\end{aligned}\right\}\tag{12}$$

and

$$\left.\begin{aligned}\left(\sum\sigma_i^*\varphi_{i1}\right)l+\left(\sum\sigma_i^*\varphi_{i6}\right)m+\left(\sum\sigma_i^*\varphi_{i5}\right)n=X_s\\\left(\sum\sigma_i^*\varphi_{i6}\right)l+\left(\sum\sigma_i^*\varphi_{i2}\right)m+\left(\sum\sigma_i^*\varphi_{i4}\right)n=Y_s\\\left(\sum\sigma_i^*\varphi_{i5}\right)l+\left(\sum\sigma_i^*\varphi_{i4}\right)m+\left(\sum\sigma_i^*\varphi_{i3}\right)n=Z_s\end{aligned}\right\}\tag{13}$$

Taking note of Eqn. (2), Eqns. (12) ~ (13) can be obtained directly from the equilibrium equations and the boundary conditions of static force under geometrical representation, provided we make the representation conversion of Eqn. (2) on stress. But one of virtue of the equilibrium equations under mechanical representation is that the stress which holds the equilibrium equations certainly holds the compatibility equation.

4 FUNDAMENTAL EQUATIONS OF EIGEN ELASTIC MECHANICS

Although the equilibrium Eqn. (12) deduced above are expressed with the modal stress, they do not break away from the geometrical form, and is not easy to calculation. Considering the representation conversion Eqn. (2), the equilibrium Eqn. (12) can be still written as the form of tensor.

$$\sigma_{ik,kj}+\sigma_{jk,ki}=0\tag{14}$$

Due to symmetry on (i, j) in two sides, it can be written as

$$\Delta\sigma=0\tag{15}$$

Δ is a symmetrical differential operator matrix of the second order, and called stress differential operator matrix,

$$\Delta=\begin{bmatrix}\Delta_{11}&0&0&0&\Delta_{31}&\Delta_{21}\\0&\Delta_{22}&0&\Delta_{32}&0&\Delta_{21}\\0&0&\Delta_{33}&\Delta_{32}&\Delta_{31}&0\\0&\Delta_{33}&\Delta_{32}&(\Delta_{22}+\Delta_{33})&\Delta_{21}&\Delta_{31}\\\Delta_{31}&0&\Delta_{31}&\Delta_{21}&(\Delta_{11}+\Delta_{33})&\Delta_{32}\\\Delta_{21}&\Delta_{21}&0&\Delta_{31}&\Delta_{32}&(\Delta_{11}+\Delta_{22})\end{bmatrix}\tag{16}$$

where $\Delta_{ij}=\Delta_{ji}=\partial^2/\partial x_i\partial x_j$

The equilibrium Eqn. (15) can be written as the operationalized form under mechanical representation[5~7].

$$\Delta_i^*\sigma_i^*=0\qquad i=1,2,\cdots,6\tag{17}$$

They are six independent scalar equations, in which the stress operator is $\Delta_i^*=\varphi_i^{\mathrm{T}}\Delta\varphi_i$. The compatibility equation of elastic body can also be written as[5~7].

$$\nabla_i^*\varepsilon_i^*=0\qquad i=1,2,\cdots,6\tag{18}$$

∇_i^* is the strain operator, $\nabla_i^*=\varphi_i^{\mathrm{T}}\nabla\varphi_i$. $[\nabla]$ is also a symmetrical differential operator matrix of the second order.

$$[\nabla]=\begin{bmatrix}0&\nabla_{33}&\nabla_{22}&-\nabla_{23}&0&0\\\nabla_{33}&0&\nabla_{11}&0&-\nabla_{13}&0\\\nabla_{22}&\nabla_{11}&0&0&0&-\nabla_{12}\\-\nabla_{23}&0&0&-\nabla_{11}/2&\nabla_{12}/2&\nabla_{13}/2\\0&-\nabla_{13}&0&\nabla_{12}/2&-\nabla_{22}/2&\nabla_{23}/2\\0&0&-\nabla_{12}&\nabla_{13}/2&\nabla_{23}/2&-\nabla_{33}/2\end{bmatrix}\tag{19}$$

where $\nabla_{ij}=\nabla_{ji}=\partial^2/\partial x_i\partial x_j$.

There exist the following Hooke's form between the stress operator and the strain operator.

$$\nabla_i^*=\lambda_i\Delta_i^*\qquad i=1,2,\cdots,6\tag{20}$$

It is seen from Eqns. (17), (18) and (20) that the equilibrium equation and the compatibility equation of elastic body under mechanical representation are indistinguishable.

5 DEFINITE EQUATIONS OF EIGEN ELASTIC MECHANICS

Define the stress function under mechanical representation as[2]

$$\sigma_i^* = \nabla_i^* \phi_i \quad i = 1, 2, \cdots, 6 \tag{21}$$

Substituting Eqn. (21) into Eqn. (17) and Eqn. (13) respectively, we get the definite equations under mechanical representation for anisotropic elastic mechanics.

$$\left.\begin{array}{l} \Delta_i^* \nabla_i^* \phi_i = 0 \quad i = 1, 2, \cdots, 6 \\ \sum \nabla_i^* \phi_i(\varphi_{i1} l + \varphi_{i6} m + \varphi_{i5} n) = X_s \\ \sum \nabla_i^* \phi_i(\varphi_{i6} l + \varphi_{i2} m + \varphi_{i4} n) = Y_s \\ \sum \nabla_i^* \phi_i(\varphi_{i5} l + \varphi_{i4} m + \varphi_{i3} n) = Z_s \end{array}\right\} \tag{22}$$

It is seen from Eqn. (22) that the solution of elastic mechanics under mechanical representation is, in fact, the results of the modal superimposition.

$$\sigma = \sigma_1^* \varphi_1 + \sigma_2^* \varphi_2 + \cdots + \sigma_6^* \varphi_6 = \sum \sigma^{(i)} \tag{23}$$

where $\sigma^{(1)}$ is an approximate value of the first order, and $\sigma^{(2)}, \cdots, \sigma^{(6)}$ are the revised value of the second to the 6th order respectively. The solution will be accurate after it is revised six time. In engineering calculation, considering only some former modal solutions, the needing precision can be satisfied.

6 APPLICATION

It does not loss generality to take the plane problem of isotropic elastic body as an example and give the solving process of eigen elastic mechanics. More important thing is that the result of new theory should go back to the classical one in the particular case.

There are two subspaces in isotropic elastic body. So, the structure of the mechanical space is

$$W = W_1^{(1)}(\varphi_1) \oplus W_2^{(2)}(\varphi_2, \varphi_3) \tag{24}$$

where $\varphi_1 = \sqrt{2}[1, 1, 0]^T/2$, $\varphi_2 = \sqrt{2}[1, -1, 0]^T/2$, $\varphi_3 = [0, 0, 1]^T$.

The strain operator of the plane problem of elastic mechanics is

$$\nabla_1^* = \sqrt{2}(\nabla_{11} + \nabla_{22})/2, \quad \nabla_2^* = \sqrt{(\nabla_{11} - \nabla_{22})^2/2 + \nabla_{12}} \tag{25}$$

Substituting Eqn. (25) into Eqn. (22), and using Eqn. (20),

$$(\nabla_{11} + \nabla_{22})^2 \phi_1 = 0, \quad [(\nabla_{11} - \nabla_{22})^2 + 4\nabla_{12}^2]\phi_2 = 0 \tag{26}$$

Open up Eqn. (26), and using the property of differentiation, $\nabla_{11}\nabla_{22} = \nabla_{12}^2$, Eqn. (26) degrades. It is just the double harmonic equation familiarized in elastic mechanics. Thus, the eigen elastic mechanics go back to the classical elastic mechanics for the plane problem of isotropic elastic body

$$\nabla^2 \nabla^2 \phi = 0 \tag{27}$$

where ϕ is just Airy function, and we have

$$\sigma_x = \partial^2 \phi/\partial y^2, \quad \sigma_y = \partial^2 \phi/\partial x^2, \quad \tau_{xy} = -\partial^2 \phi/\partial x \partial y \tag{28}$$

For example, let $\phi = \tau xy$, we have $\sigma_1^* = \sigma_x + \sigma_y = 0$, $\sigma_2^* = \tau_{xy} = -\tau$. They are just the pure shear case.

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