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Eigen theory of elastic mechanics for anisotropic solids ①

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Abstract: Eigen characters of the fundamental equations, equilibrium equation of stress and harmony equation of deformation, of the traditional elastic mechanics under geometrical space were testified by means of the concept of standard space, and the modal equilibrium equation and the modal harmony equation under mechanical space were obtained. Based on them and the modal Hooke's law, a new system of the fundamental equation of elastic mechanics is given. The advantages of the theory given here are as following: the form of the fundamental equation is in common for both isotropy and anisotropy, both force method and displacement method, both force boundary and displacement boundary; the number of stress functions is equal to that of the anisotropic subspaces, which avoids the man-made mistakes; the solution of stress field or strain field is given in form of the modal superimposition, which makes calculation simplified greatly; no matter how complicated the anisotropy of solids may be, the complete solutions can be obtained.

Key words: anisotropic solid; elastic mechanics; eigen form; standard space **Document code: A**

1 INTRODUCTION

The traditional theory of elastic mechanics is composed of three fundamental equations, which is equilibrium equation, harmony equation, and generalized Hooke's law. The former two equations hold for all solid materials under the condition of continuity, but the latter is related to the properties of solids. The task of elastic mechanics is trying to obtain the solution of these closing equations and corresponding boundaries. But up to now, we find that the solving capability of elastic mechanics is very limited, especially in solution of the anisotropic problem, which make us have to seek the help of some numerical methods, such as FEM. When introspecting this phenomenon, it is found that there are some inadquacy or draw-backs in the traditional elastic mechanics. The author think that the traditional elastic mechanics has the geometric draw-back, because it describes the fundamental equations under geometrical image. In other words, the geometrical space restricts the universality of physical equations, which is brought by certain result of the geometrical mechanics. It is known that the Hooke's law under geometrical image does not present the mechanical properties of solids clearly, it is related to coordinates. That is, elasticity and anisotropy of solids are totally put into an elastic matrix. But in fact, the anisotropy of solids is related to coordinates, elasticity is not. As a result, the constitutive equation is succinct, but it makes great burden on physical equations because of introducing geometrical factors into them, which makes the solution of elastic mechanics very difficult. This paper gives up the method of the synthetic elastic matrix under geometrical image, and studies the process of elastic mechanics under mechanical image. The results show that the fundamental equations in the form of tensor are changed into the form of scalar, which makes the solution easy and simple, especially for anisotropic elastic mechanics.

The idea of eigen elasticity originated from the works[1,2] of Kelvin, after quieted nearly a hundred years, it was raised again in 1980's, and improved[3~5]. Based on it, the author developed a standard space theory[6~11], which is much beneficial to the solution of anisotropic elastic mechanics. This paper tries to study elastic mechanics in mechanical space.

2 EIGEN PROPERTY OF SOLID MATERIALS

The matrix form of the generalized Hooke's law under geometrical image is[3~5]

$$\sigma = \boldsymbol{C}\varepsilon \tag{1}$$

it holds the eigen equation of elasticity

$$(\boldsymbol{C} - \lambda \boldsymbol{I})\varphi = 0 \tag{2}$$

where λ and φ are eigenvalue and eigenvector of elastic coefficients matrix $\boldsymbol{C}$, respectively. The former is eigen elastic module (Kelvin module), and is not related to coordinates; the latter is standard space, and indicates the anisotropic direction of solids. Thus, the elastic coefficients matrix under geometrical image can be decomposed spectrally under mechanical image, that is

$$\boldsymbol{C} = \Phi \Lambda \Phi^{\mathrm{T}} \tag{3}$$

where Φ is eigen modal matrix, and is orthogonal and symmetric. Λ is eigen elastic matrix, and is diagonal.

So, the generalized Hooke's law under standard space becomes the normal form of

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$$\sigma_i^* = \lambda_i \varepsilon_i^* \qquad i = 1, 2, \cdots, 6 \tag{4(a)}$$

or

$$\sigma^* = \Lambda \varepsilon^* \tag{4(b)}$$

They are called the modal Hooke's law, in which the modal stress vector and modal strain vector are respectively

$$\sigma^* = \Phi^{\mathrm{T}} \sigma \tag{5}$$

$$\varepsilon^* = \Phi^{\mathrm{T}} \varepsilon \tag{6}$$

Eqn.(4) are six independent ones. So, under mechanical image, the mechanical properties of solids can be described by six common scalar Hooke's law.

3 EIGEN EXPRESS OF FUNDAMENTAL EQUATIONS

3.1 Eigen express of equilibrium equation

Under geometrical image, the equilibrium equation of solids in which body force is neglected is

$$\sigma_{ik'k} = 0 \tag{7(a)}$$

or

$$\sigma_{jk'k} = 0 \tag{7(b)}$$

Differentiating Eqn.(7(a)) with j and Eqn.(7(b)) with i, then adding them and yielding

$$\sigma_{ik'kj} + \sigma_{jk'ki} = 0 \tag{8}$$

It is another form of equilibrium equation, and can be written in the form of matrix due to symmetry on (i, j) in its two sides, that is

$$\Delta \sigma = 0 \tag{9}$$

in which Δ is a symmetrical differential operator matrix of order of two, and called stress differential operator matrix.

$$\Delta = \begin{bmatrix} \Delta_{11} & 0 & 0 & 0 & \Delta_{31} & \Delta_{21} \\ & \Delta_{22} & 0 & \Delta_{32} & 0 & \Delta_{21} \\ & & \Delta_{33} & \Delta_{32} & \Delta_{31} & 0 \\ & & & (\Delta_{22} + \Delta_{33}) & \Delta_{21} & \Delta_{31} \\ & \text{symmetry} & & & (\Delta_{11} + \Delta_{33}) & \Delta_{32} \\ & & & & & (\Delta_{22} + \Delta_{11}) \end{bmatrix} \tag{10}$$

where $\Delta_{ij} = \Delta_{ji} = \partial^2 / \partial x_i \partial x_j$.

Substituting Eqn.(5) into Eqn.(9), multiplying its two sides with the transpose of eigen modal matrix and using the orthogonal symmetrical property of the matrix, i.e. $\Phi^{\mathrm{T}} \Phi = \boldsymbol{I}$, it is obtained:

$$\Phi^{\mathrm{T}} \Delta \Phi \sigma^* = 0 \tag{11}$$

The author proved[11] that there exists same standard space in stress differential operator matrix Δ and elastic coefficients matrix $\boldsymbol{C}$. So, it is obtained:

$$\Phi^{\mathrm{T}} \Delta \Phi = \Delta^* \tag{12}$$

where Δ^* is diagonal and called eigen stress differential operator matrix.

Thus, under mechanical image, the equilibrium Eqn.(7) becomes

$$\Delta_i^* \sigma_i^* = 0 \qquad i = 1, 2 \cdots 6 \tag{13}$$

They are also six independent scalar equations.

3.2 Eigen express of harmony equation

Under geometrical image, the harmony equation of deformation of solids is

$$\varepsilon_{ij'kl} - \varepsilon_{kj'li} + \varepsilon_{kl'ij} - \varepsilon_{il'kj} = 0 \tag{14}$$

Because of symmetry on (i, j) in ε_{ij} and (k, l) in $\varepsilon_{ij'kl}$, Eqn.(14) can be written as

$$\nabla \varepsilon = 0 \tag{15}$$

where ∇ is also symmetrical differential operator matrix of order two, and called strain differential operator matrix.

$$\nabla = \begin{bmatrix} 0 & \nabla_{33} & \nabla_{22} & -\nabla_{23} & 0 & 0 \\ & 0 & \nabla_{11} & 0 & -\nabla_{13} & 0 \\ & & 0 & 0 & 0 & -\nabla_{12} \\ & & & -\nabla_{11} & \nabla_{12} & \nabla_{13} \\ & \text{symmetry} & & & -\nabla_{22} & \nabla_{23} \\ & & & & & -\nabla_{33} \end{bmatrix} \tag{16}$$

where $\nabla_{ij} = \nabla_{ji} = \partial^2 / \partial x_i \partial x_j$.

Comparing Eqn.(9) with Eqn.(15), and using Eqn.(1), it is obtained

$$\Delta = \nabla \boldsymbol{C} \tag{17}$$

Substituting Eqn.(3) and Eqn.(12) into Eqn.(17), it becomes

$$\nabla = \Phi \Delta^* \Phi^{\mathrm{T}} \Phi \Lambda \Phi^{\mathrm{T}} = \Phi \Delta^* \Lambda \Phi^{\mathrm{T}} \tag{18}$$

Letting $\nabla^* = \Delta^* \Lambda$, it is obtained

$$\nabla^* = \Phi^{\mathrm{T}} \nabla \Phi \tag{19}$$

Eqn.(19) shows that the strain differential operator has same form of spectrum as the stress differential operator under mechanical image, it is called eigen strain differential operator matrix.

Thus, the harmony equation of deformation under mechanical image can be written as following by using Eqn.(15) and Eqn.(19)

$$\nabla_i^* \varepsilon_i^* = 0 \qquad i = 1, 2, \cdots 6 \tag{20}$$

4 FUNDAMENTAL EQUATION OF ELASTIC MECHANICS UNDER MECHANICAL IMAGE

It is known from the equilibrium Eqn.(7) that stress tensor σ_{ij} which holds Eqn.(7) may consist of a symmetrical tensor g_{ij} of order of two:

$$\sigma_{ij} = e_{imr} e_{jns} g_{rs'mn} \tag{21}$$

where g_{rs} is just stress function.

If written in the form of vector, Eqn.(21) becomes

$$\sigma = \nabla g \tag{22}$$

Further, Eqn.(22) can be written as the eigen form under mechanical image by using Eqn.(5) and Eqn.(18) as

$$\sigma^* = \nabla^* g^* \tag{23(a)}$$

or

$$\sigma_i^* = \nabla_i^* g_i^* \qquad i = 1, 2, \cdots, 6 \tag{23(b)}$$

where $g^* = \Phi^{\mathrm{T}} g$, is modal stress function. Eqn.(23) is just operational form of the modal stress.

In the same way, the operational form of the modal strain can also be gotten:

$$\varepsilon_i^* = \Delta_i^* f_i^* \qquad i = 1, 2, \cdots, 6 \tag{24}$$

where f_i^* is modal strain function.

Now, we deduce the fundamental equation of elastic mechanics under mechanical image.

Comparing the eigen equilibrium Eqn.(13) with the eigen harmony Eqn.(20), and using the eigen Hooke's law, Eqn.(4), it is gotten

$$\nabla_i^* = \lambda_i \Delta_i^* \qquad i = 1, 2, \cdots, 6 \tag{25}$$

It shows that the eigen strain differential operator is in direct proportion to the eigen stress differential operator, the proportional coefficient is just eigen elasticity.

In order to obtain the fundamental equation, substituting Eqn.(23) and Eqn.(24) into Eqn.(13) and Eqn.(20) respectively, we get

$$\Delta_i^* \nabla_i^* g_i^* = 0 \qquad i = 1, 2, \cdots, 6 \tag{26}$$

$$\nabla_i^* \Delta_i^* f_i^* = 0 \qquad i = 1, 2, \cdots, 6 \tag{27}$$

Because of exchangeable property of the eigen differential operator, let

$$\Box_i^* = \Delta_i^* \nabla_i^* = \nabla_i^* \Delta_i^* \qquad i = 1, 2, \cdots, 6 \tag{28}$$

Thus, it is gotten

$$g_i^* = f_i^* = \psi_i \qquad i = 1, 2, \cdots, 6 \tag{29}$$

So, ψ_i can be understood as the mechanical function, it expresses not only stress, but also strain.

$$\sigma_i^* = \nabla_i^* \psi_i \qquad i = 1, 2, \cdots, 6 \tag{30}$$

$$\varepsilon_i^* = \Delta_i^* \psi_i \qquad i = 1, 2, \cdots, 6 \tag{31}$$

As a result, the fundamental equation of elastic mechanics becomes the unitized and solitary form:

$$\Box_i^* \psi_i = 0 \qquad i = 1, 2, \cdots, 6 \tag{32(a)}$$

or

$$\sum_{i=1}^{6} \Box_i^* \psi_i = 0 \tag{32(b)}$$

It is seen from above analysis that under mechanical image, distinguishing not only between stress solution and strain solution (the force method and the displacement method), but also between equilibrium equation and harmony equation are all meaningless because of existence of Eqn.(4) and Eqn.(25), and distinguishing between force boundary and displacement boundary is also meaningless because of the existence of Eqns.(30) ~ (32). Therefore, they are much beneficial to the solution of elastic mechanics. Another important thing of Eqn.(32) is that it points out that the number of independent mechanical functions is equal to that of anisotropic subspaces, which makes the choice of mechanical function of various anisotropy much convenient, and is completely different from the method of the traditional elastic mechanics.

5 FORM OF SOLUTION AND BOUNDARY CONDITION UNDER MECHANICAL IMAGE

In order to obtain the solution of Eqn.(32) for various anisotropic solids, it is necessary to give a set of boundary condition to the fundamental equation.

Using Eqn.(5) and Eqn.(30), it is gotten

$$\nabla_i^* \psi_i = \varphi_i^{\mathrm{T}} \sigma \tag{33}$$

It also holds on the boundary of solids. Thus, the equations which have the definite solution for anisotropic solids are given as followings:

$$\nabla_i^* \Delta_i^* \psi_i = 0 \qquad i = 1, 2, \cdots, 6 \tag{34}$$

$$\nabla_i^* \psi_i \mid_s = \varphi_i^T \sigma \mid_s \qquad i = 1, 2, \cdots, 6 \tag{35}$$

where Eqn.(35) are boundary conditions under various anisotropic subspaces

In fact, Eqns.(34) and (35) only give the solution of order i, the complete solution of stress field and strain field should be superimposition of them. By Eqn.(5) and Eqn.(6), we have

$$\sigma = \varphi_1 \nabla_1^* \psi_1 + \cdots + \varphi_6 \nabla_6^* \psi_6 \tag{36}$$

$$\varepsilon = \varphi_1 \Delta_1^* \psi_1 + \cdots + \varphi_6 \Delta_1^* \psi_6 \tag{37}$$

The superimposed form of solution makes the calculation simplified because of the flexible choice of calculating order. For example, we can neglect the high order results for some problems by using the cutting technology of mode, which not only reduces the calculating work, but also still holds the necessary accuracy. The cutting error can be estimated by following Eqn.[7]

$$e_p = \frac{\lfloor 1/\lambda_{p+1} + \cdots + 1/\lambda_6 \rfloor}{\lfloor 1/\lambda_1 + 1/\lambda_2 + \cdots + 1/\lambda_6 \rfloor} \times 100\,\% \tag{38}$$

where e_p is the cutting error of order p.

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