# MAXIMUM ORTHOGONAL COMPLEMENT LIKELIHOOD ESTIMATION FOR VARIANCE COMPONENTS①

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**ABSTRACT** Based on the orthogonal complement likelihood function, the estimation formula for variance components was derived, and the Helmert's estimation formula was proved to be a special form under some assumed conditions. Finally, as an application example, the results solved by the above two formulas to a triangulateration geodetic network were shown.

**Key words** orthogonal complement likelihood function variance component estimation formula

## 1 INTRODUCTION

In order to obtain accurately the most proper weights of different types of observation, the posterior estimation methods for variance components corresponding to those have been usually applied[1]. After thorough research, the theory and approaches of estimation for variance components have been perfected[2-7]. By summarizing the presented variance components estimation formulas, it is found that only Helmert's estimation formula is strict, and other different estimation formulas are approximative ones based on that.

It is well known that, when observation errors as the stochastical variables are normally distributed, the maximum likelihood estimation for variance of unit weight is biased, but the maximum orthogonal complement likelihood is unbiased[8]. Based on this function, the author of this paper has derived the maximum orthogonal complement likelihood estimation equation for variance components, and proved that the Helmert's formula is a special one of those under some assumed conditions. It is shown by a triangulateration geodetic network that Helmert's solution of variance components is only a couple of positive real solution in the maximum orthogonal complement likelihood estimation equations for variance components.

## 2 HELMERT'S ESTIMATION FORMULA FOR VARIANCE COMPONENTS

The general linear model with two types of observation which are assumed to be uncorrelated is given by

$$\boldsymbol{V}=A\boldsymbol{X}-\boldsymbol{l} \tag{1}$$

or

$$\begin{pmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \boldsymbol{X} - \begin{pmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \end{pmatrix} \tag{2}$$

with

$$D(\boldsymbol{l})=D=\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} = Q_1\sigma_1^2+Q_2\sigma_2^2 \tag{3}$$

and

$$Q_1=\begin{pmatrix} p_1^{-1} & 0 \\ 0 & 0 \end{pmatrix};\ Q_2=\begin{pmatrix} 0 & 0 \\ 0 & P_2^{-1} \end{pmatrix} \tag{4}$$

where $A$, $A_1$, $A_2$ respectively denote the known $n \times t$, $n_1 \times t$, $n_2 \times t$ (where $n$ equals $n_1 + n_2$) designed matrices, in which the first matrix is assumed to be of full column rank, $\boldsymbol{X}$ the $t \times 1$ vector of unknown parameters, $\boldsymbol{l}$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ respectively the $n \times 1$, $n_1 \times 1$, $n_2 \times 1$ vectors of observations, $\boldsymbol{V}$, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ respectively the $n \times 1$, $n_1 \times 1$, $n_2 \times 1$ residual vectors of observations, $D$, $D_1$, $D_2$ respectively the $n \times n$, $n_1 \times n_1$, $n_2 \times n_2$ diagonal variance matrices for $\boldsymbol{l}$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, $p_1$, $p_2$ respectively the $n_1 \times n_1$, $n_2 \times n_2$ primary weight matrices for $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, which is assumed to be positive definite, $\sigma_1^2$, $\sigma_2^2$ re-

① Supported by The China National Nonferrous Metals Industry Corporation; Received Oct. 4, 1994

spectively the variance components of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$.

Usually, before least squares adjustment is applied to observation equation (1) or (2), $D_1$ and $D_2$ are unknown. Thus $P_1$ and $P_2$ aren't obtained by strict computation, but are only estimated by the posterior methods from repeated adjustment computations.

Assuming the variance of unit weight to be $\sigma_0^2$, then if $\sigma_1^2=\sigma_2^2=\sigma_0^2$, it is shown that the primary weight matrices $P_1$, $P_2$ are the most proper. The least squares estimator vector of $\boldsymbol{X}$ can be expressed as

$$\hat{\boldsymbol{X}}=N^{-1}A^TP\boldsymbol{l} \tag{5}$$

where $N=A^TPA$ (6)

with $P=\begin{pmatrix}P_1 & 0\\ 0 & P_2\end{pmatrix}$ (7)

Substituting (5) into (2) yields

$$\boldsymbol{V}_1=(A_1N^{-1}A_1^TP_1-I_1)\boldsymbol{l}_1+A_1N^{-1}A_2^TP_2\boldsymbol{l}_2 \tag{8}$$

$$\boldsymbol{V}_2=A_2N^{-1}A_1^TP_1\boldsymbol{l}_1+(A_2N^{-1}A_2^TP_2-I_2)\boldsymbol{l}_2 \tag{9}$$

where $I_1$, $I_2$ respectively denote $n_1\times n_1$, $n_2\times n_2$ identity matrices. By considering to (3) and (4), according to the law of error propagation, we can get Eqs. (10)~(13) from (8) and (9)

$$D(\boldsymbol{V}_1)=(P_1^{-1}-2A_1N^{-1}A_1^T+A_1N^{-1}N_1N^{-1}A_1^T)\sigma_1^2+A_1N^{-1}N_2N^{-1}A_1^T\sigma_2^2 \tag{10}$$

$$D(\boldsymbol{V}_2)=A_2N^{-1}N_1N^{-1}A_2^T\sigma_1^2+(P_2^{-1}-2A_2N^{-1}A_2^T+A_2N^{-1}N_2N^{-1}A_2^T)\sigma_2^2 \tag{11}$$

with $N_1=A_1^TP_1A_1$; $N_2=A_2^TP_2A_2$ (12)

and $N=N_1+N_2$ (13)

By considering $E(\boldsymbol{V}_1)=0$ and $E(\boldsymbol{V}_2)=0$, from the expectation theorem of quadratic forms, we have

$$E(\boldsymbol{V}_1^TP_1\boldsymbol{V}_1)=\mathrm{tr}(P_1D(\boldsymbol{V}_1))=[n_1-2\mathrm{tr}(N^{-1}N_1)+\mathrm{tr}(N^{-1}N_1)^2]\sigma_1^2+\mathrm{tr}(N^{-1}N_1N^{-1}N_2)\sigma_2^2 \tag{14}$$

$$E(\boldsymbol{V}_2^TP_2\boldsymbol{V}_2)=\mathrm{tr}(P_2D(\boldsymbol{V}_2))=\mathrm{tr}(N^{-1}N_1N^{-1}N_2)\sigma_1^2+[n_2-2\mathrm{tr}(N^{-1}N_2)+\mathrm{tr}(N^{-1}N_2)^2]\sigma_2^2 \tag{15}$$

If letting coefficients in (14) and (15) be

$$S_1=n_1-2\mathrm{tr}(N^{-1}N_1)+\mathrm{tr}(N^{-1}N_1)^2 \tag{16}$$

$$S_2=n_2-2\mathrm{tr}(N^{-1}N_2)+\mathrm{tr}(N^{-1}N_2)^2 \tag{17}$$

$$S_0=\mathrm{tr}(N^{-1}N_1N^{-1}N_2) \tag{18}$$

$$W_1=\boldsymbol{V}_1^TP_1\boldsymbol{V}_1 \tag{19}$$

$$W_2=\boldsymbol{V}_2^TP_2\boldsymbol{V}_2 \tag{20}$$

then the estimation equations for $\sigma_1^2$, $\sigma_2^2$ from (14) and (15) become

$$\begin{pmatrix}S_1 & S_0\\ S_0 & S_2\end{pmatrix}\begin{pmatrix}\hat{\sigma}_1^2\\ \hat{\sigma}_2^2\end{pmatrix}=\begin{pmatrix}W_1\\ W_2\end{pmatrix} \tag{21}$$

which is the Helmert's estimation equations for two variance components. If the solution $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$ of (21) don't equal $\sigma_0^2$, then the new value for the weight matrices of $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ may be computed from $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ respectively, a repeated adjustment computation is needed until $\hat{\sigma}_1^2=\hat{\sigma}_2^2=\hat{\sigma}_0^2$. When $\hat{\sigma}_1^2=\hat{\sigma}_2^2=\hat{\sigma}_0^2$, the following equality may be proved from (14) and (15)

$$E(\boldsymbol{V}^TP\boldsymbol{V})=E(\boldsymbol{V}_1^TP_1\boldsymbol{V}_1)+E(\boldsymbol{V}_2^TP_2\boldsymbol{V}_2)=(S_1+S_2+2S_0)\sigma_0^2=(n-t)\sigma_0^2 \tag{22}$$

Thus, the unbiased estimator for $\sigma_0^2$ is

$$\hat{\sigma}_0^2=\frac{\boldsymbol{V}^TP\boldsymbol{V}}{n-t} \tag{23}$$

## 3 MAXIMUM ORTHOGONAL COMPLEMENT LIKELIHOOD ESTIMATION FOR VARIANCE COMPONENTS

When $\boldsymbol{l}$ is a normally distributed observations vector, the likelihood function of $\boldsymbol{l}$ with unknown parameters vector $\boldsymbol{X}$ and unknown variance components $\sigma_1^2$, $\sigma_2^2$ is given by

$$L(\boldsymbol{l}/\boldsymbol{X},\sigma_1^2,\sigma_2^2)=(2\pi)^{-n/2}(\det D)^{-1/2}\times\exp[-(\boldsymbol{l}-A\boldsymbol{X})^TD^{-1}(\boldsymbol{l}-A\boldsymbol{X})/2] \tag{24}$$

If the observations vector $\boldsymbol{l}$ is transformed[7], then the orthogonal complement likelihood function can be obtained

$$L(\boldsymbol{l}/\sigma_1^2,\sigma_2^2)\propto(\det D\det\overline{N})^{-1/2}\times\exp[-(\boldsymbol{l}-A\hat{\boldsymbol{X}})^TD^{-1}(\boldsymbol{l}-A\hat{\boldsymbol{X}})/2]=(\det D\det\overline{N})^{-1/2}\exp(-\boldsymbol{V}^TD^{-1}\boldsymbol{V}/2) \tag{25}$$

with

$$\overline{N}=A^TD^{-1}A=(A_1^T,A_2^T)\begin{pmatrix}\sigma_1^{-2}P_1 & 0\\ 0 & \sigma_2^{-2}P_2\end{pmatrix}\begin{pmatrix}A_1\\ A_2\end{pmatrix}=N_1\sigma_1^2+N_2\sigma_2^2 \tag{26}$$

Taking the natural logarithm on both

sides, (25) leads to

$$\ln L(\boldsymbol{l}/\sigma_1^2,\ \sigma_2^2) \propto -(1/2)\ln(\det D) - 1/2 \ln(\det \overline{N}) - (1/2)\boldsymbol{V}^T D^{-1}\boldsymbol{V} \quad (27)$$

From (3) we have

$$\boldsymbol{V}^T D^{-1}\boldsymbol{V} = (\boldsymbol{V}_1^T,\ \boldsymbol{V}_2^T)\begin{pmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{V}_1 \\ \boldsymbol{V}_2\end{pmatrix} = \boldsymbol{V}_1^T P_1 \boldsymbol{V}_1 \sigma_1^{-2} + \boldsymbol{V}_2^T P_2 \boldsymbol{V}_2 \sigma_2^{-2} \quad (28)$$

To maximize $L(\boldsymbol{l}/\sigma_1^2,\ \sigma_2^2)$, its partial derivatives with respect to $\sigma_1^2$ and $\sigma_2^2$ are equated to zero. At first, differentiating with respect to $\sigma_1^2$ on the both sides of (27) and equating that to zero, we can get

$$\frac{\partial \ln L(\boldsymbol{l}/\sigma_1^2,\ \sigma_2^2)}{\partial \sigma_1^2} = -1/2\left[\frac{\partial \ln(\det D)}{\partial \sigma_1^2} + \frac{\partial \ln(\det \overline{N})}{\partial \sigma_1^2} + \frac{\partial \boldsymbol{V}^T D^{-1}\boldsymbol{V}}{\partial \sigma_1^2}\right.$$

$$= -\mathrm{tr}(D^{-1}Q_1)/2 + \mathrm{tr}(\overline{N}^{-1}N_1)\cdot\sigma_1^{-4}/4 + \boldsymbol{V}_1^T P_1 \boldsymbol{V}_1 \sigma_1^{-4}$$

$$= -n_1\sigma_1^{-2}/2 + \mathrm{tr}(\overline{N}^{-1}N_1)\cdot \sigma_1^{-4}/2 + \boldsymbol{V}_1^T P_1 \boldsymbol{V}_1 \sigma_1^{-4} = 0 \quad (29)$$

Expanding $\overline{N}$ in a Neumann series at $N\sigma_1^{-2}$ and taking the zero-and first-order terms of the series expansion yields

$$\begin{aligned}\overline{N}^{-1} &= (N_1\sigma_1^{-2} + N_2\sigma_2^{-2})^{-1} \\ &= [(N\sigma_1^{-2} + (N_1\sigma_1^{-2} + N_2\sigma_2^{-2} - N\sigma_1^{-2})]^{-1} \\ &= N^{-1}\sigma_1^2 - N^{-1}\sigma_1^2(N_1\sigma_1^{-2} + N_2\sigma_2^{-2} - N\sigma_1^{-2})N^{-1}\sigma_1^2 \\ &= 2N^{-1}\sigma_1^2 - N^{-1}N_1N^{-1}\sigma_1^2 - N^{-1}N_2N^{-1}\sigma_1^4\sigma_2^{-2} \quad (30)\end{aligned}$$

Substituting (30) into (29) and reducing, we get

$$S_1\hat{\sigma}_1'^2 + S_0\hat{\sigma}_1'^4\hat{\sigma}_2'^{-2} = W_1 \quad (31)$$

Similarly, differentiating with respect of $\sigma_2^2$ on the both sides of (27) and equating that to zero, we get

$$S_0\hat{\sigma}_2'^4\hat{\sigma}_1'^{-2} + S_2\hat{\sigma}_2'^2 = W_2 \quad (32)$$

where $S_0$, $S_1$, $S_2$ were respectively given by (16), (17), (18). In (31) and (32), we apply $\hat{\sigma}_1'^2$ and $\hat{\sigma}_2'^2$ to express the maximum orthogonal complement estimator of variance components $\sigma_1^2$ and $\sigma_2^2$ to distinguish from their Helmert's estimators $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$.

Writing (31) and (32) into the matrix form yields

$$\begin{pmatrix} S_1 & S_0\hat{\sigma}_1'^4/\hat{\sigma}_2'^4 \\ S_0\hat{\sigma}_2'^4/\hat{\sigma}_1'^4 & S_2 \end{pmatrix}\begin{pmatrix}\hat{\sigma}_1'^2 \\ \hat{\sigma}_2'^2\end{pmatrix} = \begin{pmatrix} W_1 \\ W_2\end{pmatrix} \quad (33)$$

If assuming that the following equality were right

$$\hat{\sigma}_1'^4/\hat{\sigma}_2'^4 = \hat{\sigma}_2'^4/\hat{\sigma}_1'^4 = 1 \quad (34)$$

then (33) would become (21). But among the repeated adjustment computation, (34) is not always correct, thus (21) is an approximative maximum orthogonal complement likelihood estimate equation for variance components $\sigma_1^2$ and $\sigma_2^2$. If $\hat{\sigma}_1'^2 = \hat{\sigma}_2'^2 = \hat{\sigma}_0^2$, from (33) we may derive (22) and (23). Thus $\hat{\sigma}_1'^2$, $\hat{\sigma}_2'^2$ are also unbiased.

By changes of variables

$$\theta_1 = \hat{\sigma}_1'^2 \text{ and } \theta_2 = \hat{\sigma}_2'^2 \quad (35)$$

(33) becomes

$$S_1\theta_1\theta_2 + S_0\theta_1^2 - W_1\theta_2 = 0 \quad (36)$$

$$S_0\theta_2^2 + S_1\theta_1\theta_2 - W_2\theta_1 = 0 \quad (37)$$

Solving (36) for $\theta_2$ yields

$$\theta_2 = \frac{S_0\theta_1^2}{W_1 - S_1\theta_1} \quad (38)$$

Substituting (38) into (37) to eliminate $\theta_2$ and reducing, we have

$$\theta_1^3 + b_2\theta_1^2 + b_1\theta_1 + b_0 = 0 \quad (39)$$

with
$$b_2 = \frac{S_0S_2W_1 - S_1^2W_2}{S_0^3 - S_0S_1S_2}$$

$$b_1 = \frac{2S_1W_1W_2}{S_0^3 - S_0S_1S_2}$$

$$b_0 = \frac{-W_1^2W_2}{S_0^3 - S_0S_1S_2}$$

which is an equation of 3-th order with one variable. From (39) and (38), we can obtain three couples of solution for $\theta_1$ and $\theta_2$, in which only positive real solutions are proper. If there is no couple of positive real solutions with (39), it is shown that primary weight matrix is not proper and should be renewed.

## 4 AN ADJUSTMENT EXAMPLE

Fig. 1 shows a pentagon (in a plane) with one center point, in which the points 1 and 2 are horizontal control stations with known $\boldsymbol{x}$, $\boldsymbol{y}$ coordinates. In order to obtain accurately the $\boldsymbol{x}$, $\boldsymbol{y}$ coordinates of other four points, all the fifteen possible angles and nine possible

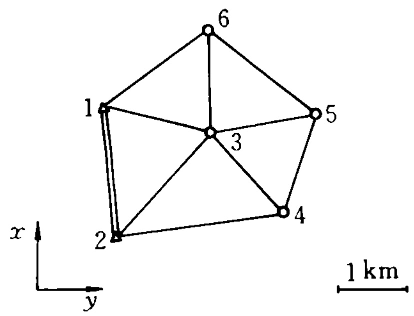


**Fig. 1 The Triangulateration geodetic network**

distances had been measured. The prior variance of the angle observation is $\hat{\sigma}_\beta^2 = 3.7636$ ($s^2$) obtained by the angle closing errors of triangle, and the prior variance of distance observation $\hat{\sigma}_S^2 = 1.2769$ ($cm^2$) by the closing errors of reciprocal observations. If assuming that the variance of unit weight $\sigma_0^2$ equals $\hat{\sigma}_\beta^2$, that is

$$\sigma_0^2 = \hat{\sigma}_\beta^2 = 3.7636(s^2)$$

then the primary weights of angle and distance observation respectively are $P_{1i} = 1$ (for $i = 1, 2\cdots, 15$) and $P_{2j} = 2.9474(s^2/cm^2)$ (for $j = 1, 2, \cdots, 9$). The changes of variance components are obtained from equations (21) and (38), (39) through iterations. These results are given in Table 1 for making a comparison. where $c$ denotes the number of repeated adjustment computation. Table 1 shows that because the primary weights of angle and distance observation is proper, among the repeated adjustment computations there are always a couple solution for $\theta_1$ and $\theta_2$ which are positive real(see the data underscored).

**Table 1 Changes of variance components solved by the two kinds of equations**

| $c$ | Helmert | | Maximum orthogonal complement likelihood | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\hat{\sigma}_1^2$ | $\hat{\sigma}_2^2$ | $\theta_1$ | | | $\theta_2$ | | |
| | | | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 2.6578 | 1.9980 | <u>2.2456</u> | −10.6669 | 3.1679 | <u>2.5223</u> | 3.1313 | −22.1476 |
| 2 | 3.7727 | 3.7433 | <u>3.7763</u> | −3.1143 | 5.8140 | <u>3.7861</u> | 4.6359 | −40.1194 |
| 3 | 3.7645 | 3.7615 | <u>3.7632</u> | −14.3187 | 5.8082 | <u>3.7636</u> | 4.6067 | −40.0801 |
| 4 | 3.7637 | 3.7634 | | | | | | |

One can also see the results solved by the two methods are all converged to given $\sigma_0^2$.

## REFERENCES

1 Helmert F R. Die Ausgleichungsrechnung nach der Methode der Kleinsten Quadrate, 3. Auflage, Teubner. Leipzig, 1924.

2 Kubik K. Schatzung der Gewichte der Fehlergleichungen beim Ausgleichungsproblem nach vermittelnden Beobachtungen, 1967, 92(3): 173—178.

3 Kubik K. Schatzfunktionen fur Varianzen, Kovarianzen und andere Parameter in Ausgleichungsaufgaben. Dissertation Wien, 1967.

4 Welsch W. A posteriopi Varianzschatzung nach Helmert, 1978, AVN 85(2): 55—63.

5 Koch K R. Schatzung von Varianzkomponenten, Allg. Vermessungsnach-richten, 1979, 85: 264—269.

6 Forstner W. Ein Verfohren zur Schatzung von Varianz-Kovarianz-Komponenten. 1979, AVN 86: 446—453.

7 Grafarend E W. Statistics and Decisions, 1985, Suppl 2: 407—441.

8 Koch K R. Manuscripta Geodaetica, 1987, 309—313.

**(Edited by He Xuefeng)**