A NEW ORTHOTROPIC YIELD FUNCTION DESCRIBABLE ANOMALOUS BEHAVIOR OF MATERIALS[①]

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ABSTRACT

A new non-quadratic orthotropic yield function describable anomalous behavior of materials has been put forward, which like the yield function raised by Hill recently, has no limitations possessed by the previous similar functions. It is simple and clear, especially, all the materials constants involved can be determined using only unaxial tension test. Therefore, this function is more convenient to use. The application of this function to 1100 aluminium sheets indicates that the 1100 aluminium alloy is a material possessing anomalous behavior and the reasonable power value of its yield function is about 1.68. Most deviations between the calculated values using this function and the experimental data of flow stresses are smaller than 4 percent.

Key words: orthotropy anomalous behavior non-qudratic yield function

1 INTRODUCTION

Hill's orthotropic quadratic yield function [1] overestimated the influence of anisotropy of materials and could not be used for those materials possessing so-called "anomalous behavior". Since the 1970s there have been some non-quadratic yield functions advanced successly, and up till now four functions among that can be used for these materials. However the yield function advanced by Bassani[2] and the case IV of new yield function by Hill in 1979[3] could only be used for the case that the sheet metal is planar isotropic. In order to accomodate the planar anisotropy of a sheet metal, recently Hill put forward another yield function [4]:

$$|\sigma_1+\sigma_2|^m+\left(\frac{\sigma}{\tau}\right)^m|\sigma_1-\sigma_2|^m + |\sigma_1^2+\sigma_2^2|^{(m/2)-1}[-2a(\sigma_1^2-\sigma_2^2) + b(\sigma_1-\sigma_2)^2\cos2\theta]\cos2\theta = (2\sigma)^m$$

where σ_1 and σ_2 are the two in-plane principal stresses; θ is the angle between the direction of the larger principal stress and the rolling direction of the sheet metal; σ is the flow stress in equibiaxial tension state ($\sigma_1=\sigma_2$); τ is the flow stress in pure shear parallel to the orthotropic axes ($\theta=45°$); a and b are constants determined by σ, τ and the flow stresses in uniaxial tension state. It is obvious that the above formula is not simple, especially since at least tests under three stress states must be made to determine the material constants involved in it. Thereby it is not convenient for use.

The Fourth stated yield function was proposed by Montheillet *et al* recently[5], but the material constants involved in their function were determined by assuming that the material anisotropy is four-fold symetric. Thus, it is difficult to say that, the planar anisotropy of a sheet metal can be correctly reflected. Moreover, their function can not yet be used for the case of planar isotropy.

2 THE NEW YIELD FUNCTION AND ITS CORRESPONDING EQUATIONS

2.1 *The Form of the New Yield Function*

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Adopting both the characteristics of the yield function proposed by the author[6] and the yield function proposed by Montheillet *et al*, the form of the newly proposed yield function in this paper is:

$$f = c[(\sigma_x+a\sigma_y)^2]^{m/2}+h[(\sigma_x-\sigma_y)^2+b\tau_{xy}^2]^{m/2} = \sigma_i^m \qquad (1)$$

where x and y stand for the two in-plane anisotropic principal axial directions and are generally believed to be the rolling direction and the transverse direction of the sheet; m is the power value of the yield function; c, a, h and b are anisotropic parameters; σ_i is the effective stress. For convenience we can take the flow stress σ_0 in unaxial tension along the rolling direction of the sheet as σ_i, then we have:

$$c + h = 1 \qquad (2)$$

Therefore, the material constants needed to be determined total four.

2.2 *Corresponding Flow Rule*

2.2.1 The Flow Rule in the Anisotropic Principal Axes Coordinate System

Taking the yield function as a plastic potential function and using Druker's axiom, we can obtain the flow rule corresponding to the yield function:

$$\left.\begin{aligned}
d\varepsilon_x &= m d\lambda\{c(\sigma_x + a\sigma_y)[(\sigma_x + a\sigma_y)^2]^{(m-2)/2} \\
&\quad + h(\sigma_x - \sigma_y)[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{(m-2)/2}\} \\
d\varepsilon_y &= m d\lambda\{ca(\sigma_x + a\sigma_y)[(\sigma_x + a\sigma_y)^2]^{(m-2)/2} \\
&\quad - h(\sigma_x - \sigma_y)[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{(m-2)/2}\} \\
d\varepsilon_t &= -(d\varepsilon_x + d\varepsilon_y) \\
&= - m d\lambda\{c(1 + a)(\sigma_x + a\sigma_y) \\
&\quad \times [(\sigma_x + a\sigma_y)^2]^{(m-2)/2}\} \\
d\gamma_{xy} &= m d\lambda b h \tau_{xy}[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{(m-2)/2}
\end{aligned}\right\} \qquad (3)$$

$$d\varepsilon_i = m d\lambda \sigma_i^{m-1} = m d\lambda \sigma_0^{m-1} \qquad (4)$$

where $d\varepsilon_t$ is strain increment of sheet thickness; $d\varepsilon_i$ is effective strain increment; $d\lambda$ is a positive coefficient related with strain hardening level.

From equation (4), we have:

$$m d\lambda = d\varepsilon_i/\sigma_i^{m-1} = d\varepsilon_i/\sigma_0^{m-1} \qquad (5)$$

As for the effective stress σ_i, from equations (1) and (2), we have:

$$\begin{aligned}
\sigma_i &= \sigma_0 \\
&= \{c(\sigma_x + a\sigma_y)^2]^{m/2} \\
&\quad + h[(\sigma_x - \sigma_y)^2 + b\tau_{xy}^2]^{m/2}\}^{1/m}
\end{aligned} \qquad (6)$$

2.2.2 Flow Rule in the Principal Stresses Coordinate System

Adopting the above definitions of σ_1, σ_2 and θ and using χ to express the ratio of σ_2/σ_1, i.e.

$$\chi = \sigma_2/\sigma_1 \quad (\sigma_1 > \sigma_2) \qquad (7)$$

then the familiar stress transformation formulas and strain transformation formulas can be written out as:

$$\left.\begin{aligned}
\sigma_x &= \sigma_1\cos^2\theta + \sigma_2\sin^2\theta \\
&= (\frac{1+\chi}{2} + \frac{1-\chi}{2}\cos 2\theta)\sigma_1 \\
\sigma_y &= \sigma_1\sin^2\theta + \sigma_2\cos^2\theta \\
&= (\frac{1+\chi}{2} - \frac{1-\chi}{2}\cos 2\theta)\sigma_1 \\
\tau_{xy} &= -(\sigma_2 - \sigma_1)\sin\theta\cos\theta \\
&= (\frac{1-\chi}{2}\sin 2\theta)\sigma_1
\end{aligned}\right\} \qquad (8)$$

$$\left.\begin{aligned}
d\varepsilon_1 &= \frac{d\varepsilon_x + d\varepsilon_y}{2} \\
&\quad + \frac{d\varepsilon_x - d\varepsilon_y}{2}\cos 2\theta + \frac{1}{2}d\gamma_{xy}\sin 2\theta \\
d\varepsilon_2 &= \frac{d\varepsilon_x + d\varepsilon_y}{2} \\
&\quad - \frac{d\varepsilon_x - d\varepsilon_y}{2}\cos 2\theta - \frac{1}{2}d\gamma_{xy}\sin 2\theta \\
d\gamma_{12} &= (d\varepsilon_x - d\varepsilon_y)\sin 2\theta - d\gamma_{xy}\cos 2\theta
\end{aligned}\right\} \qquad (9)$$

Substituting equation (8) into equation (6), we have:

$$\begin{aligned}
\sigma_i &= \sigma_0 \\
&= \frac{1}{2}[cA^m + hB^m(1-\chi)^m]^{1/m}\sigma_1
\end{aligned} \qquad (10)$$

where

$$\left.\begin{aligned}
A &= (1+\chi)(1+a) + (1-\chi)(1-a)\cos 2\theta \\
B &= 2(\cos^2 2\theta + \frac{b}{4}\sin^2 2\theta)^{1/2}
\end{aligned}\right\} \qquad (11)$$

Substituting equation (3) into equation (9), and introducing equations (4), (8), (10) and (11), and rearranging, we have:

$$\left.\begin{aligned}
d\varepsilon_i &= \frac{2[cA^m + hB^m(1-\chi)^m]^{(m-1)/m}}{c[(1+a) + (1-a)\cos 2\theta]A^{m-1} + hB^m(1-\chi)^{m-1}}d\varepsilon_1 \\
&= \frac{2[cA^m + hB^m(1-\chi)^m]^{(m-1)/m}}{c[(1+a) - (1-a)\cos 2\theta]A^{m-1} - hB^m(1-\chi)^{m-1}}d\varepsilon_2 \\
&= -\frac{[cA^m + hB^m(1-\chi)^m]^{(m-1)/m}}{c(1+a)A^{m-1}}d\varepsilon_t \\
&= \pm\frac{[cA^m + hB^m(1-\chi)^m]^{(m-1)/m}d\gamma_{12}}{[c(1-a)A^{m-1} - (1-b)hB^{m-2}(1-\chi)^{m-1}\cos 2\theta]\sin 2\theta}
\end{aligned}\right\} \qquad (12)$$

Because $d\varepsilon_i$ is positive, the sign of the last formula of equation (12) is determined by $d\gamma_{12}$. In addition for the cases of $\theta = 0°$ and $\theta = 90°$, $d\gamma_{12} = 0$, the last formula of equation (12) becomes an infinitive, so $d\varepsilon_i$ in these cases cannot be determined from $d\gamma_{12}$.

2.3 *Formulae of $d\varepsilon_i$ Expressed by Strain*

Increment Components

From equation (3), we have:

$$d\varepsilon_x+d\varepsilon_y=md\lambda c(1+a)[(\sigma_x+a\sigma_y)^2]^{(m-1)/2}$$
$$\times\{[d\varepsilon_x-d\varepsilon_y-\frac{1-a}{1+a}(d\varepsilon_x+d\varepsilon_y)]^2$$
$$+\frac{(2d\gamma_{xy})^2}{b}\}^{1/2}$$
$$=md\lambda 2h[(\sigma_x-\sigma_y)^2+b\tau_{xy}^2]^{(m-1)/2}$$

Substituting equation (6) into equation (4), and replacing the stress items by strain increments using the above equation, and rearranging, we have:

$$d\varepsilon_t=\{\frac{1}{c^{1/(m-1)}}[(\frac{d\varepsilon_x+d\varepsilon_y}{1+a})^2]^{m/2(m-1)}$$
$$+\frac{1}{h^{1/(m-1)}}[(\frac{d\varepsilon_x-d\varepsilon_y}{2}-\frac{1-a}{1+a}\frac{d\varepsilon_x+d\varepsilon_y}{2})^2$$
$$+\frac{(d\gamma_{xy})^2}{b}]^{m/2(m-1)}\}^{(m-1)/m} \tag{13}$$

It can be seen that when $a=1$, and $d\varepsilon_x$ and $d\varepsilon_y$ are principal strain increments ($d\gamma_{xy}=0$), i.e. when the sheet metal is planar isotropic, equation (13) reduces to the equation of case IV of the new yield function raised by Hill in 1979.

Using the strain transformation formula, equation (13) can be transformed into a formula expressed by $d\varepsilon_1$, $d\varepsilon_2$ and $d\gamma_{12}$ or principal strain increments.

2.4 *Determination of Material Constants*

2.4.1 Determination By Plastic Strain Ratio

For the case of uniaxial tension, $x=0$. Substituting $x=0$ into equations (12) and (11), and according to the definition, the plastic strain ratio r is:

$$r_\theta=-\frac{c[(1+a)-(1-a)\cos 2\theta]A^{m-1}-hB^m}{2c(1+a)A^{m-1}} \tag{14}$$

$$A=(1+a)+(1-a)\cos 2\theta \quad (\chi=0) \tag{11a}$$

Substituting $\theta=0, 45, 90°$ into equation (14) and introducing equation (2), we have:

$$\left.\begin{aligned}
r_0&=-\frac{ca-h}{c(1+a)}=\frac{1}{c(1+a)}-1\\
r_{45}&=-\frac{c(1+a)^m-hB^{m/2}}{2c(1+a)^m}\\
&=-\frac{c(1+a)^m-(1-c)b^{m/2}}{2c(1+a)^m}\\
r_{90}&=-\frac{ca^{m-1}-h}{c(1+a)a^{m-1}}\\
&=-\frac{c(1+a^{m-1})-1}{c(1+a)a^{m-1}}
\end{aligned}\right\} \tag{15}$$

From the first formula of equation (15), we have:

$$c=\frac{1}{(1+a)(1+r_0)} \tag{16}$$

Substituting equation (16) into the third formula of equation (15) and rearranging, we have:

$$a^{m-1}=\frac{r_0+a(1+r_0)}{ar_{90}+1+r_{90}} \tag{17}$$

If $m=2$, from equation (17) we have $a=r_0/r_{90}$. If $m\neq 2$, we can obtain a using a step by step calculating method, and the initial value of a can be taken to be r_0/r_{90}.

From the second formula of equation (15), we have:

$$b=[\frac{c(1+a)^m(1+2r_{45})}{1-c}]^{2/m} \tag{18}$$

Therefore, for a certain m value, we can calculate the values of a, c and b on the basis of the experimentally measured values of r_0, r_{45} and r_{90}.

For the cases of planar isotropy, i.e. $r_0=r_{45}=r_{90}=r$, from equations (16)~(18) we have

$c=1/[2(1+r)]$,

$a=1$, $b=4$

2.4.2 Determination By Flow Stresses

The above material constants can also be determined using flow stresses. But, in this case where none of plastic strain ratios is used, at least one experimental datum of flow stresses that can reflect the normal anisotropic behavior of the sheet metal is necessary to be used besides those data of uniaxial flow stresses, because the latters cannot reflect the normal anisotropy. The former may be the flow stress of a sheet metal under uniaxial compression state along the sheet thickness or equibiaxial tension state.

For uniaxial tension tests, substituting $\sigma_t=\sigma_0$ and $\chi=0$ into equations (10) and (11) and rearranging, we have:

$$\sigma_\theta=\frac{2\sigma_0}{\{c[(1+a)+(1-a)\cos 2\theta]^m+h(4\cos^2 2\theta-b\sin^2 2\theta)^{m/2}\}^{1/m}} \tag{19}$$

When $\theta=45°$ and $90°$, we have:

$$\left.\begin{aligned}
(\sigma_0/\sigma_{90})^m&=ca^m+h=1-c(1-a^m)\\
(2\sigma_0/\sigma_{45})^m&=c(1+a)^m+hb^{m/2}\\
&=c(1+a)^m+(1-c)b^{m/2}
\end{aligned}\right\} \tag{20}$$

For equibiaxial stress tension tests, substituting $\chi=1$ into equations (10) and (11), and using σ_b to express equibiaxial tension flow stress, we have:

$$(\sigma_0/\sigma_b)^m=c(1+a)^m \tag{21}$$

From equations (20) and (21), we have:

$$\left.\begin{aligned} a &= \{1 - [1 - (\sigma_0/\sigma_{90})^m](1 + a)^m \\ &\quad \div (\sigma_0/\sigma_{b})^m\}^{1/m} \\ c &= (\sigma_0/\sigma_{b})^m/(1 + a)^m \\ b &= \left[\frac{(2\sigma_0/\sigma_{45})^m - c(1 + a)^m}{1 - c}\right]^{2/m} \end{aligned}\right\} \quad (22)$$

Here the step by step calculating method is also used to calculate a, and then calculate c and b.

2.5 *Check on the Usability of the Yield Function*

Substituting equations (18) and (16) into the second formula of equation (20), we have:

$$(2\sigma_0/\sigma_{45})^m = 2(1 + a)^{m-1}(1 + r_{45}) \div (1 + r_0) \quad (23)$$

It can be seen that when the sheet metal is planar isotropic, i. e. $r_0 = r_{45} = r_{90} = r$, $a = 1$ and $\sigma_0 = \sigma_{45} = \sigma_{90}$, m may be an any number determined by the experimental data, thereby, this function is usable.

Substituting equation (16) into equation (21), we have:

$$(\sigma_{b}/\sigma_0)^m = (1 + r_0)/(1 + a)^{m-1} \approx (1 + r)/2^{m-1} \quad (24)$$

Hence, for those materials with $r < 1$, under proper m values, equation (24) can fit the cases of $\sigma_{b}/\sigma_0 > 1$, consequently, this yield function is usable to materials possessing "anomalous behavior".

2.6 *Determination of Power Value m*

From equation (23), we have:

$$m = \ln\left[\frac{2(1 + r_{45})}{(1 + a)(1 + r_0)}\right]/\ln\left[\frac{2\sigma_0}{(1 + a)\sigma_{45}}\right] \quad (25)$$

Hence, for certain experimental data, m and a can be calculated by combining equations (25) and (17). But, the calculated results using this equation are greatly influenced by errors of individual experimental datum. The practice proved that the better method is to calculate m values by regression analysis using a large number of experimental data.

It should be noted that it is difficult for r value to have an accuracy higher than ± 0.1, as pointed out by Kusnierz[7], which means that for the materials with $r < 1.0$, the errors of r values are generally larger than 10%. Therefore, it is not appropriate to calculate m values completely depending on r values; and the m values should be determined by means of the flow stresses and the total level of r value $r_{cp}[= (r_0 + 2r_{45} + r_{90})/4]$, just like the following example.

For the materials with planar isotropy, equation (25) becomes an infinite, thereby, the m values should be calculated using equation (24), i. e.

$$m = \ln[2(1 + r)]/\ln(2\sigma_{bi}/\sigma_0) \quad (25a)$$

3 APPLICATION EXAMPLE

Take the 1100 aluminium sheet described in ref. [5] as an example, whose related properties are listed in Tables 1 and 2. In Tables 1 and 2, $\bar{\varepsilon} = 2/[\sqrt{3}\ln(t_0/t)]$. The samples with different thickness were cut and rolled out from a same cold-rolled sheet of thickness t_0.

According to the test method described in ref. [5], although $\bar{\varepsilon} \neq \varepsilon_e$, the deformation levels under the same $\bar{\varepsilon}$ be identical. In Table 2 σ_{p_1} and σ_{p_2} are the flow stresses of the sheet metal under the plane strain tension state of $\theta = 0°$, $d\varepsilon_y = 0$ and $\theta = 90°$, $d\varepsilon_x = 0$ respectively. The flow stress is σ_c under uniaxial compression state along the sheet thickness. According to the assumption that the spherial stress tensor does not influence the yield state of material, σ_c may be adopted as the flow stress σ_{b} of the sheet metal under equibiaxial tension stress state.

Table 1 Experimental data of r

$\bar{\varepsilon}$	r_0	r_{45}	r_{90}	r_{cp}
0.00	0.77	1.08	0.75	0.92
0.26	0.60	0.91	0.78	0.80
0.52	0.59	0.85	0.69	0.75

Table 2 Parameters involved in $\sigma = \sigma_s + K\bar{\varepsilon}^n$

	σ_0	σ_{45}	σ_{90}	σ_c	σ_{p_1}	σ_{p_2}
σ_s/MPa	51.4	48.3	53.9	56.1	62.2	65.0
K/MPa	82.6	81.9	83.1	79.5	89.8	91.1
n	0.472	0.481	0.417	0.423	0.408	0.375

In Table 1 r values without consistent change rule indicate that these experimental data are not all very accurate. Therefore this paper adopts equation (22) to determire the values of c, a and b using some m-values, and calculate the r_{cp}-values from these values. That m -value by which the calculated r_{cp} -value is in keeping with the experimental data was taken as the reasonable m -value for this deformation level, see Table 3.

Table 3 Calculation of reasonable m value of 1100 aluminium sheet

ε	0			0.26			0.52		
m	1.70	1.674	1.60	1.70	1.671	1.60	1.70	1.688	1.60
r_{cp}	0.96	0.92	0.81	0.84	0.80	0.70	0.77	0.75	0.64

From Table 3, it is reasonable to take the mean of 1.674, 1.671 and 1.688, i.e. 1.68, as the general m value of of 1100 aluminium sheet. This value is 12 percent larger than that given by Montheillet *et al*.

Whether the above results are reliable can be examined by comparing the experimental data and the predicated values of plane strain tension flow stresses.

Assuming that χ_p is the ratio of two principal stress ratio in the case of plane strain tension, and according to the condition $d\varepsilon_2 = 0$, from the second formula of equation (12) we can obtain:

$$c[(1+a)-(1-a)\cos 2\theta][\frac{1+\chi_p}{1-\chi_p}(1+a) + (1-a)\cos 2\theta]^{m-1} - hB^m = 0$$

Then assuming

$$P = \frac{hB^m}{c[(1+a)-(1-a)\cos 2\theta]} = [\frac{1+\chi_p}{1-\chi_p}(1+a)+(1-a)\cos 2\theta]^{m-1} \tag{26}$$

the χ_p value solved from equation (26) is

$$\chi_p = 1 - \frac{2(1+a)}{P^{1/(m-1)} - (1-a)\cos 2\theta + 1 + a} \tag{27}$$

Substituting equation (26) into equation (11), then substituting equation (11) and equation (27) into equation (10), and rearranging, we have:

$$\frac{\sigma_p}{\sigma_0} = \frac{P^{1/(m-1)} - (1-a)\cos 2\theta + 1 + a}{(1+a)[cP^{m/(m-1)} + h(4\cos^2 2\theta + b\sin^2 2\theta)^{m/2}]^{1/m}} \tag{28}$$

Substituting $\theta = 0°$ and $90°$ into equation (28) respectively, we can obtain the predicated values of σ_{p_1}/σ_0 and σ_{p_2}/σ_0, whose relative deviations compared with the experimental data are listed in Table 4. It can be seen that it is appropriate to take $m = 1.68$, and under this condition, most relative deviations are smaller than 4.0 percent, except a few are larger than 5.0 percent.

Table 4 Relative deviations between the predicated and experimental data of plane strain tension flow stresses

ε	0		0.26		0.52	
m	1.68	1.50	1.68	1.50	1.68	1.50
a	0.8251	0.8460	0.8510	0.8682	0.8855	0.8986
c	0.3110	0.3166	0.3193	0.3559	0.3218	0.3596
b	5.0110	5.0727	4.7635	4.8111	4.6222	4.6620
Δ_1%	−3.12	−4.95	−3.92	−5.82	−2.82	−4.82
Δ_2%	−3.71	6.26	−5.47	−7.91	−4.00	−6.10

4 CONCLUSIONS

(1)The yield function proposed in this paper is applicable. It can not only be used for common anisotropic materials, but also can be used for materials possessing anomalous behavior. Moreover, this function is the most simple, universal and conveniecnt function of its kind.

(2)The 1100 aluminium sheet belongs to materials possessing anomalous behavior, and the power value of its yield function is about 1.68.

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